Infrared Problem in QCD Revisited

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Abstract

Infrared divergencies in Large N QCD are eliminated resulting in explicit perturbation expansion for mass ratios in terms of universal effective coupling constant $\alpha = 1$.Zeroth approximation correspond to Bessel roots and higher terms are all calculable exactly, without any approximations.

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1 Introduction

Large N QCD ([1]) is a remarkable mathematical problem, having most desirable universality and at the same time being close enough to reality. Being solved with proper generality, it may serve as a basis for quantitative theory of hadrons, by means of systematic expansion in inverse powers of N_c^2. However, all attempts to do so in the past met the brick wall of infrared divergencies.

In most simple terms, the infrared divergency is the fact that spectrum of the theory is discrete whereas in perturbation theory it is continuous. There are freely flying interacting quarks and gluons, which all must get confined so that physical spectrum consists instead of infinitely rising Regge trajectories of mesons and glueballs.

Everybody believes in this picture now, though it was never rigorously proven. The real question is how to transform the sum of planar graphs of Large N QCD into some sensible expansion for hadronic spectrum. This must be doable, as the effective coupling at hadron scale is not so large, and hadrons are not far from some bag model of free quarks. The spatial bag is of course unacceptable, as it totally screws the particle spectrum being non-relativistic and transitionally non-invariant.

It was almost 35 years ago that I attempted to solve this problem and made some advances ([2],[3]). I developed systematic method of approximating the meromorphic function like 2-point function of gauge invariant composite fields in Large N QCD as a sum of infinite number of Pole terms with positive residues. For such an approximation to be unique it has to have higher than powerlike convergence at large Euclidean momenta – otherwise moving poles or changing residues would be allowed without changing leading powerlike term as long as the net change in power terms decreases faster than this asymprotic term, say $\log(t)$ with $t = p^2$ being usual momentum squared in Minkowski space.

There are theorems in Pade theory [4] which guarantee that so called Stieltjes-function (analytic in cut plane with positive discontinuity across the cut) preserve this property in any order of approximation. The poles of Pade approximant are all located along the cut with positive residues – so that discontinuity reduces to finite sum of positive δ terms. This Stieltjes-function property generalizes to arbitrary matrix functions of one variable in which case it becomes equivalent to one particle unitarity + analyticity. In the limit when number of particles goes to infinity this property is just what we need in Large N QCD.

So, the limit of infinite number of poles with fixed positions, introduced and studied in [2] represents the correct framework for large N confining theory regardless of its asymprotic properties at large Euclidean momenta. In case of asymptotically conformal (or asymptotically free) theory the Pade regularization further simplifies and produces explicit calculable results depending of values of operator dimensions (normal or anomalous).

In this paper we briefly summarize this theory and advance it further, producing explicit terms of perturbation expansion of QCD mass ratios in terms of calculable terms of perturbation expansion of the (matrix of) 2-point functions of conformal fields. We use dimensional regularization which fits nicely into our framework, and supplement it by infrared regularization using Matrix Pade theory in the limit of infinite number of poles. Important step is that we are able to eliminate the infrared cutoff in every order in dimensionless effective coupling α, normalized so that it must be set to 1 after summation of perturbation expansion.

This is continuation of old work [2, 3], but unlike that old work, now we produce analytic rather than numerical formulas. Given planar graphs for the matrix of 2-point function, which are universal functions of ϵ times powers of $\lambda t^{-\frac{\epsilon}{2}}$ where λ is t'Hoofts coupling constant and $d = 4 - \epsilon$ is dimension of space.

Note that at any positive $\epsilon < 3$ we still expect confinement to hold, as it is known for $\epsilon = 1, 2$. The tricky limit of $\epsilon \to +0$ will be considered later. It is important that one does it after our Pade regularization, in which case all divergencies miraculously disappear from perturbation theory.

Yes, we are simply presenting infinity-free perturbation expansion for Large N QCD observables with calculable coefficients and nice physical properties (zeroth approximation corresponds to masses proportional to roots of Bessel functions in agreement with experiment). Terms of our expansion correspond to planar graphs, but after Pade regularization and dimensional transmutations all cutoffs disappear and we get universal numbers just as it should be. There is no ambiguity nor approximations in computing these numbers, do not get misled by the words "Pade approximation". We start with approximation, but later take the limit returning us to original theory after renormalizing coupling constant. The infrared cutoff R describing spacial scale of the infrared regularization is tend to ∞ term by term in perturbation expansion (to be more precise this limit corresponds to effective coupling constant $\alpha \to 1$).

2 Pade approximation in Hilbert Space

Let us consider some CFT perturbed by some set of soft operators such as mass terms. We shall further assume that this is confining $N_c = \infty$ theory, with only planar graphs left and discrete spectrum of masses rising all the way to infinity to match CFT asymptotics in deep Euclidean region of momenta. In this limit we know very important property of the infinite matrix G_{IJ} of 2-point functions of bilinear quark operators $\hat{O}_J(x)$: free particle unitarity + analyticity. This is meromorphic matrix function of the form:

$$G_{IJ}(p^2) = \int d^d x e^{ipx} \left\langle \hat{O}_I(0), \hat{O}_J(x) \right\rangle = \sum_i \frac{Z_i}{m_i^2 - p^2} \Psi_I^i \Psi_J^{\dagger i}. \tag{1}$$

We are working with functions of single variable p^2 (in Minkowski metric) assuming that kinematical factors depending on direction of momentum are involved in definition of states I, J. So, in general the state $|I\rangle$ depends upon $n_\mu = \frac{p_\mu}{|p|}$ and the 2-point function $G_{IJ}(p^2)$ is an irreducible tensor built of n_μ. For example, for conserved vector currents $\hat{O}_J = \overline{\psi}\gamma_\mu\psi$ there must be tensor $\delta_{\mu\nu} - n_\mu n_\nu$. In higher order operators $\overline{\psi}\Gamma_A \nabla_{\alpha_1} \nabla_{\alpha 2} ... \nabla_{\alpha_n} \psi$ with Γ_A being one of 16 independent matrices for Dirac spinors, there will be multiple invariant tensor terms in 2-point function, depending on n_μ. We shall ignore these details at this general stage of discussion, leaving them for the next Section where we switch to QCD. Unitarity implies that matrix spectral density

$$\rho(t) = \Im G(t + i0), \tag{2}$$

vanishes outside of the positive axis where it reduces to infinite number of positive definite pole terms

$$\rho(t) = \pi \sum_i Z_i \delta\left(t - m_i^2\right) \Psi^i \otimes \Psi^{\dagger i}, Z_i > 0, m_i^2 >= 0. \tag{3}$$

The matrix positivity property of the spectral function

$$\langle a \,|\rho(t)|\, a\rangle >= 0, \tag{4}$$

for arbitrary state $|a\rangle$ as well as absence of singularities outside positive axis in complex t plane is characteristic of so called Stieltjes functions. The discrete spectrum is not necessary for a function to belong to Stieltjes class, for example $\sqrt{-t}$ is a Stieltjes function. More up to a point, the perturbative spectral density in any order of naive perturbation expansion is also a Stieltjes function. We expect it to remain such function beyond perturbation theory, but the continuum spectrum condense to a discrete one.

In the leading order of any CFT, including of course the free quark-gluon theory, the operators of different dimension do not correlate. There are growing conformal families of operators with the same dimension, corresponding to eigenstates of dilatation operator. In general case, beyond leading approximation, the matrix of 2-point functions will become non-diagonal, so that more general version of Pade theory must be employed to guarantee unitarity + analyticity in whole Hilbert space of infinite number of free mesons.

This theory ([4]) is a straightforward matrix generalization of one-dimensional Pade theory. The Pade approximant is essentially a continued fraction, which is obtained by sequence of transformations

$$\begin{aligned} G_n(\xi) &= A_n \frac{1}{1 - \xi G_{n+1}(\xi)} A_n^\dagger, && (5)\\ \xi &= \left(1 + \frac{t}{\Lambda}\right), && (6)\\ G(t) &= G_0\left(\xi\right), && (7)\\ A_n A_n^\dagger &= G_n(0); \end{aligned}$$

The constant matrix A_n is defined up to irrelevant right multiplication by unitary matrix, for a positive Hermitian $G_n(0)$ in euclidean region $t = -\Lambda$ in terms of its eigenvectors $\langle a\,|i\,\rangle$and positive eigenvalues g_i

$$\begin{aligned} \langle a\,|G_n(0)|\,b\rangle &= \sum_i \langle a\,|i\rangle\, g_i\, \langle i\,|b\rangle\,, && (8)\\ \langle a\,|A_n|\,i\rangle &= \langle a\,|i\rangle \sqrt{g_i}. && (9) \end{aligned}$$

Iterating these transformations forward we obtain the continued fraction

$$G_0\left(\xi\right) = A_0 \frac{1}{1 - \xi A_1 \frac{1}{1 - \xi A_2 \frac{1}{1 - \dots} A_2^\dagger} A_1^\dagger} A_0^\dagger; \tag{10}$$

Let us invert 5

$$G_{n+1}(\xi) = \frac{1}{\xi}\left(1 - A_n^\dagger \frac{1}{G_n(\xi)} A_n\right). \tag{11}$$

In order to find the value of $G_{n+1}(0)$ we need to expand $G_n(\xi) \to G_n(0) + \xi G_n^{'}(0)$ and we find

$$G_{n+1}(0) = A_n^\dagger \frac{1}{G_n(0)} G_n^{'}(0) \frac{1}{G_n(0)} A_n, \tag{12}$$

This way we can recursively find these coefficients uniquely from the Taylor expansion of original function $G(t) = G_0(\xi)$.

The continued fraction will have $\frac{P}{Q}$ form with polynomial numerator and denominator determined from recurrent equations

$$\begin{aligned} G_n(\xi) &= P_n(\xi)\frac{1}{Q_n(\xi)}, && (13)\\ A_n A_n^\dagger &= G_n(0); && (14)\\ Q_{n+1}(\xi) &= \frac{1}{A_n} P_n(\xi), && (15)\\ P_{n+1}(\xi) &= \frac{1}{A_n}\frac{P_n(\xi) - G_n(0)Q_n(\xi)}{\xi} && (16) \end{aligned}$$

Truncation of continued fraction to $\left[\frac{N}{N}\right]$ approximant corresponds to setting

$$\begin{aligned} Q_{N+1}(\xi) &= 1, && (17)\\ P_{N+1}(\xi) &= 0 && (18) \end{aligned}$$

and iterating equations backwards to P_0, Q_0 the same equation in its inverse form:

$$\begin{aligned} P_n(\xi) &= A_n Q_{n+1}(\xi),\\ Q_n(\xi) &= \frac{1}{G_n(0)}\left(P_n(\xi) - \xi A_n P_{n+1}(\xi)\right). \end{aligned}$$

In the $\frac{P}{Q}$ form the conservation of positivity is not immediately clear, but in original form 5 we may prove it as follows. By taking discontinuity at ξ^+ at positive real $\xi > 1$ of inverse relation 11 we find

$$\begin{aligned} \Im G_{n+1}(\xi^+) &= \frac{1}{\xi} A_n^\dagger \frac{1}{G_n(\xi^-)} \Im G_n(\xi^+) \frac{1}{G_n(\xi^+)} A_n, && (19)\\ \left\langle \alpha \left| \Im G_{n+1}(\xi^+) \right| \alpha \right\rangle &= \left\langle \beta \left| \Im G_n(\xi^+) \right| \beta \right\rangle >= 0, && (20)\\ |\beta\rangle &= \frac{1}{\sqrt{\xi}} \frac{1}{G_n(\xi^+)} A_n \left|\alpha\right\rangle && (21) \end{aligned}$$

Thus, by induction, all matrix functions $G_n(\xi)$ starting with original one at $n = 0$ are Stieltjes matrix functions. Moreover, we can prove by induction that all matrix functions $G_n(\xi)$ are Hermitian matrices at real $\xi < 1$, corresponding to Euclidean region of momenta. Takin Hermitian conjugate of 11 and assuming that $G_n(\xi)$ is Hermitian we see that the same is true about $G_{n+1}(\xi)$:

$$\begin{aligned} G^\dagger_{n+1}(\xi) &= \frac{1}{\xi}\left(1 - A^\dagger_n \frac{1}{G^\dagger_n(\xi)} A_n\right) \\ &= \frac{1}{\xi}\left(1 - A^\dagger_n \frac{1}{G_n(\xi)} A_n\right) \\ &= G_{n+1}(\xi). \end{aligned} \tag{22}$$

Note that there is kind of gauge invariance of Pade approximant with respect to right matrix multiplication by a constant matrix W (independent of t):

$$Q(t) \rightarrow Q(t)W, \tag{23}$$
$$P(t) \rightarrow P(t)W \tag{24}$$

As for the left matrix multiplication it results in similarity transformation for PQ^{-1} so we can use it to diagonalize matrix G at normalization point $t = -\Lambda$.We use the gauge invariance to choose convenient normalization of these P, Q below.

The continued fraction being good practical way to build finite order approximant, their general properties in the limit of large N are better studied in the form suggested by Pade himself for the general $\left[\frac{M}{N}\right]$ approximant:

$$G_0(\xi)Q_0(\xi) - P_0(\xi) = O(\xi^{N+M+1}). \tag{25}$$

Comparing coefficients at $\xi^r, r = 0, 1, ...N + M$ we get system of linear matrix equations for matrix coefficients of polynomials $P_0(\xi), Q_0(\xi)$ relating these coefficients to Taylor coefficients of $G_0(\xi)$. Namely, the last $N - 1$ equations, which do not involve P_0 (its expansion ending at ξ^M) produce linear equations for coefficients of Q_0:

$$[G_0(\xi)Q_0(\xi)]_{\{M+1,M+N\}} = 0, \tag{26}$$

where $[...]_{\{n,m\}}$ stands for part of Taylor expansion with degrees from n to m.

Adding gauge condition, say $Q_0(0) = 1$ we get unique solution for Q_0 after which P_0 can be obtained directly as

$$P_0(\xi) = [G_0(\xi)Q_0(\xi)]_{\{0,M\}} \tag{27}$$

Note that in this method we did not have to take any square roots from $G_0(\xi)$. So, is this really the same solution as we have built above using continued fraction? Let us check at the $\left[\frac{0}{1}\right]$ approximant. In continued fraction we get

$$\begin{aligned}
G_0(\xi) &= A_0 \frac{1}{1-\xi A_1 A_1^\dagger} A_0^\dagger \qquad (28) \\
&= A_0 \frac{1}{1-\xi G_1(0)} A_0^\dagger \\
&= A_0 \frac{1}{1-\xi A_0^\dagger \frac{1}{G_0(0)} G_0'(0) \frac{1}{G_0(0)} A_0} A_0^\dagger \\
&= \frac{1}{\frac{1}{A_0} - \xi A_0^\dagger \frac{1}{G_0(0)} G_0'(0) \frac{1}{G_0(0)}} A_0^\dagger \\
&= \frac{1}{\frac{1}{A_0 A_0^\dagger} - \xi \frac{1}{G_0(0)} G_0'(0) \frac{1}{G_0(0)}} \\
&= \frac{1}{\frac{1}{G_0(0)} - \xi \frac{1}{G_0(0)} G_0'(0) \frac{1}{G_0(0)}} \\
&= G_0(0) \frac{1}{1-\xi \frac{1}{G_0(0)} G_0'(0)} \\
&= P \frac{1}{Q}
\end{aligned}$$

which is the same we would have obtained much easier from matrix Pade equations:

$$\left(G_0(0) + G_0^{'}(0)\xi\right)(1+q_1\xi) - p_0 = O(\xi^2) \qquad (29)$$

Note that we could have represented the same continuos fraction differently:

$$G_0(\xi) = A_0 \frac{1}{1-\xi A_1 A_1^\dagger} A_0^\dagger \qquad (30)$$

$$= \ldots \qquad (31)$$

$$= \frac{1}{\frac{1}{G_0(0)} - \xi \frac{1}{G_0(0)} G_0'(0) \frac{1}{G_0(0)}} \qquad (32)$$

$$= \frac{1}{1-\xi G_0'(0) \frac{1}{G_0(0)}} G_0(0) \qquad (33)$$

$$= \frac{1}{\tilde{Q}} \tilde{P}. \qquad (34)$$

This is so called left matrix approximant, which is just another way to represent the same continuos fraction. The difference arises at the moment of truncation and tends to zero with order of approximation, as both approximants converge to the Stiltjes matrix function with exponential or better convergence rate. In case of meromorphic function under consideration the convergence means that positions and residues of poles converge to correct values. Interesting fact is that the Pade approximation for masses decrease monotonously with order N of approximation so they always overestimate true mass spectrum and converge to every mass from above. This remarkable property will be used below (see also [2]).

Writing Pade equation 25 as dispersion integral with matrix spectral density we find set of linear equations for Q

$$\int_0^\infty ds\,(1+s/\Lambda)^{(r-2N-1)}\,\rho(s)Q(s) = 0; r = 0, ...N, \tag{35}$$

These equations mean that $Q(s)$ is an orthogonal matrix polynomial with respect to matrix measure

$$d\sigma(s) = ds\,(1+s/\Lambda)^{(-2N-1)}\,\rho(s) \tag{36}$$

The general solution 27 for P can be rewritten as dispersion integral

$$\begin{aligned} P(t) &= [G_0(\xi)Q_0(\xi)]_{\{0,N\}} && (37)\\ &= G_0(\xi)Q_0(\xi) - [G_0(\xi)Q_0(\xi)]_{\{N+1,\infty\}} && (38)\\ &= G(t)Q(t) - \int_0^\infty \frac{ds}{\pi(s-t)}\left(\frac{(1+t/\Lambda)}{(1+s/\Lambda)}\right)^{N+1}\rho(s)Q(s). && (39) \end{aligned}$$

3 The M-limit of matrix Pade Approximant and Higher Spins

We are going to study these equations in the limit of large N, Λ. The details of this limit depend upon the spin structure of 2-point functions, which we ignored above.

Now we represent the matrix spectral density as sum of power terms with decreasing powers, coming from soft perturbations of CFT:

$$\begin{aligned} \langle I\,|\rho(s)|\,J\rangle &= s^{\nu_I}\left(\langle I\,|\sigma|\,J\rangle + \sum_K \langle I\,|g_K|\,J\rangle\, s^{-\Delta_K^{IJ}}\right), && (40)\\ \nu_I &= \Delta_I - d/2, && (41)\\ \Delta_K^{IJ} &= (\Delta_I - \Delta_J + \mu_K)/2 > 0; && (42) \end{aligned}$$

Here μ_K are mass dimensions of operators g_K . We assume that $\Delta_I >= \Delta_J$ which is always possible in virtue of symmetry of 2-point function. Therefore, expansion in K goes in negative powers of s corresponding to UV asymptotic expansion. The matrix $\hat{\sigma}$ is independent of s and is block diagonal in space of conformal operators. It is some tensor made of n_μ with simple properties.

The linear integral equation for Q can be solved exactly using Greens function satisfying ([2]):

$$\int_0^\infty dtt^\nu \left(1+t/\Lambda\right)^{(r-2N-1)} G_\nu(t,s) = s^\nu \left(1+s/\Lambda\right)^{(r-2N-1)}; r=0,...N. \quad (43)$$

Replacing the factor

$$s^{\nu_I} \left(1+s/\Lambda\right)^{(r-2N-1)} \quad (44)$$

in 35 by

$$\int_0^\infty dtt^{\nu_I} \left(1+t/\Lambda\right)^{(r-2N-1)} G_{\nu_I}(t,s) \quad (45)$$

we find

$$\int_0^\infty dt \left(1+t/\Lambda\right)^{(r-2N-1)} t^{\nu_I} S_{IM}(t) = 0, r=0,...N-1, \quad (46)$$

$$\int_0^\infty dsG_{\nu_I}(t,s) \sum_J \left(\langle I|\sigma|J\rangle + \sum_K \langle I|g_K|J\rangle t^{-\Delta_K^{IJ}} \right) \langle J|Q(s)|M\rangle = 0. \quad (47)$$

We know solutions of above equations for $G_\nu(t,s)$ and $S_{IM}(t)$ ([2]):

$$S_{IM}(t) = \oint_C \frac{d\omega}{2\pi i} f_{\nu_I}(\omega) \left(1+t/\Lambda\right)^\omega \left\langle I \left| \hat{W} \right| M \right\rangle, \quad (48)$$

$$f_\nu(\omega) = \frac{\Gamma(2N+1-\omega)\Gamma(-\omega)N^{2(1-\nu)}}{\Gamma(N+1-\nu-\omega)\Gamma(N+1-\omega)}. \quad (49)$$

$$G_\nu(t,s) = \oint_C \frac{d\omega}{2\pi i} \oint_{C'} \frac{d\omega'}{2\pi i} \frac{1}{\omega'-\omega} \frac{f_\nu(\omega)}{f_\nu(\omega')} \frac{\left(1+t/\Lambda\right)^\omega}{\left(1+s/\Lambda\right)^{\omega'}} \quad (50)$$

Here the contour C encloses the poles of $f_\nu(\omega)$ which are located at $\omega = 0,...N$ and C' encloses its zeroes, which are located at

$\omega = N+k-\nu; k=1,...\infty$. By taking residues at poles of $f_\nu(\omega)$ we observe that $G_\nu(t,s)$ is an N-degree polynomial in t with s

dependent coefficients. In the same way, $S_{IM}(t)$ is an N-degree polynomial in t, so called Jacobi polynomial. The matrix $\hat{W}$ in $S_{IM}(t)$

remains arbitrary constant matrix, but this does not lead to ambiguity, as it can be absorbed into definition of Q by means of above

gauge transformations. In virtue of this gauge invariance we may choose $\hat{W} = \hat{\sigma}$ in $S_{IM}(t)$.

So, we now have linear integral equation of general form

$$S = G\left(\hat{\sigma} + \widehat{F}\right) Q \quad (51)$$

4 The M-limit of Pade approximant as free particle Field Theory

The formulas of Pade approximant dramatically improve in the limit $N \to \infty, \Lambda \to \infty$ at fixed $R^2 = \frac{N^2}{\Lambda}$. Rather than representing numerical approximation these formulas describe some general transformation of the whole underlying field theory, amounting to placing it in a bag in some extra dimension. We shall refer to this limit as meromorphic limit or M-limit. Note that this is *different* from naive prescription to grow Λ linearly with N which one might try first. We are moving $\Lambda \sim N^2$ which is much faster. In the M-limit we can replace

$$\begin{aligned} \omega &= N^2 w, & (52)\\ (1 + t/\Lambda)^{N^2 w} &\to \exp(tR^2 w), & (53)\\ f_\nu(N^2 w) &\to (-w)^{\nu-1} \exp(-\frac{1}{w}). & (54) \end{aligned}$$

which produces the Bessel functions for $S_{IM}(t)$.

In the M-limit expansion coefficients for polynomials satisfy explicit matrix equations (see [2]). We repeat old arguments here in what I hope is cleaner and simpler form. We shall treat Taylor expansions as scalar products in space of all powers of one variable:

$$\begin{aligned} \langle I\,|Q(t)|\,J\rangle &= \overrightarrow{T}(tR^2) \cdot \langle I\,|\overrightarrow{q}|\,J\rangle\,, & (55)\\ \langle I\,|P(t)|\,J\rangle &= \overrightarrow{T}(tR^2) \cdot \langle I\,|\overrightarrow{p}|\,J\rangle\,, & (56)\\ \overrightarrow{T}(t) &= \left\{1, t, t^2, ...\right\}, & (57)\\ \overrightarrow{q} &= \{q_0, q_1, ...\}, & (58)\\ \overrightarrow{p} &= \{p_0, p_1, ...\}, & (59) \end{aligned}$$

So, our matrices q, p are now also vectors in space of powers of t in addition to being matrices in Hilbert space. The full dot $\cdot$ denotes scalar products of matrices and vectors in this space of all powers of one variable. With these notations Pade equations reduce to the following form:

$$\begin{aligned} \left(\hat{\sigma} + \widehat{F}\right) \cdot \overrightarrow{q} &= \hat{\sigma}\,\overrightarrow{q}_0, & (60)\\ \langle I\,|q_{0,n}|\,J\rangle &= \frac{(-1)^n \langle I\,|1|\,J\rangle}{n!\Gamma(\nu_J + n + 1)}, & (61) \end{aligned}$$

$$\langle I\,|F_{mn}|\,J\rangle = \frac{(-1)^m}{m!\Gamma(\nu_I + m)} \sum_K \langle I\,|g_K|\,J\rangle\, R^{2\Delta_K^{IJ}} \frac{\Gamma(\nu - \Delta_K^{IJ} + n)}{(\Delta_K^{IJ} + m - n)\Gamma(\Delta_K^{IJ} - n)}. \quad (62)$$

Note that this formula for matrix F_{mn} has the following property. The term with $K = 0$ corresponding to the leading term in spectral density can be recovered by setting $\Delta_K^{IJ} = 0$ and noting that at $m \neq n$ it vanishes due to the pole of $\Gamma(-n)$in denominator. In general case

$$\frac{1}{(m-n)\Gamma(-n)} = \delta_{nm}(-1)^m m! \tag{63}$$

Thus such term would contribute the constant $\langle I \left| g_0 \right| J \rangle$ to $\langle I \left| F_{mn} \right| J \rangle$ which is precisely the leading term $\langle I \left| \hat{\sigma} \right| J \rangle$ in spectral density. We just singled it out in our equations so that we can build perturbation expansion. The general solution can be represented as matrix inversion:

$$\langle I \left| \overrightarrow{q} \right| J \rangle = \left\langle I \left| \left(\hat{\sigma} + \widehat{F} \right)^{-1} \hat{\sigma} \right| J \right\rangle \cdot \langle J \left| \vec{q_0} \right| J \rangle \, , \tag{64}$$

with perturbation expansion simply corresponding to geometric series for inverse matrix.

The relation between P and Q also simplifies in M-limit:

$$\overrightarrow{p} = \hat{\sigma} \cdot \widehat{H} \cdot \overrightarrow{q}, \tag{65}$$

$$\langle I \left| H_{mn} \right| J \rangle = \Phi_m(\nu_I + n - m) + \sum_k \langle I \left| g_k \right| J \rangle \, R^{2\Delta_K^{IJ}} \, \Phi_m(\nu_I - \Delta_K^{IJ} + n - m), \tag{66}$$

with

$$\Phi_m(a) = -\sum_{l=0}^{m} \frac{\Gamma(l+a)}{l!} \tag{67}$$

Finally, the equation for the mass spectrum becomes operator eigenvalue problem:

$$\Psi^\dagger \left(\overrightarrow{T}(m^2 R^2) \cdot \overrightarrow{q} \right) = 0; \tag{68}$$

The leading CFT (or free quark) approximation corresponds to block-diagonal σ, ν so that the old Bessel solution

$$Q(t) \quad \rightarrow \quad u^{-\frac{\nu}{2}} I_\nu(2\sqrt{u}), \tag{69}$$

$$P(t) \quad \rightarrow \quad \frac{\sigma}{\sin(\pi\nu)} u^{\frac{\nu}{2}} I_{-\nu}(2\sqrt{u}), \tag{70}$$

$$u \quad = \quad -tR^2. \tag{71}$$

is recovered in conformal limit. After that, standard perturbation theory wisdom can be applied, including level splitting, mixing and transitions. Conformal classification of particles, valid in the zeroth approximation, will break in higher orders and degeneracy of conformal multiplets will be removed. Perturbatively, there are no fundamental problems with computation of terms, except for planar graph computation of arbitrary conformal operators. Conformal symmetry will be broken down to the Lorentz symmetry, and states will be classified accordingly, as relativistic particles, by spin and internal quantum numbers.

Let us discuss the mathematical meaning of Pade regularization as described above. First, let us compare it with conventional Pade approximation, in its matrix form. The conventional Pade approximation at any order N preserves the Stieltjes property by placing poles at positive axis and guaranteeing positivity of residues. The same is true with the matrix generalization ([4]). In perturbation expansion the Stieltjes property still holds but it degenerates to continuous spectrum of poles reproducing powerlike discontinuity with positive spectral density matrix. In this sense matrix Pade approximation gets us closer to reality than the perturbation expansion it approximates. It approximates the bad continuum spectrum by a good discrete one, but with so far incorrect masses.

Or one may say that we are approximating original QCD by taking free meson theory of infinite number of particles as an Anzatz and fitting their parameters to approximate perturbation theory as well as possible. At finite position Λ the approximants of functions with continuum spectrum were proven to converge at $N \rightarrow \infty$ absolutely in cut plane except vicinity of the real axis where the original continuous discontinuity is approximated by sum of delta functions.

In our case this is just the other way around. First of all, the true function is known to be meromorphic, so that ordinary Pade approximant with finite position Λ will approximate meromorphic function by a finite sum of poles. Such approximation is known to converge even faster. Second of all, we use the M-limit of Pade approximant where it also becomes meromorphic. So we approximate meromorphic function by meromorphic function, by varying pole positions and residues to get minimal deviations from perturbation expansion at large Euclidean momenta.

How small are these deviations? The Bessel function, corresponding to zeroth order in perturbation expansion, approaches its powerlike asymptotics at $t \rightarrow -\infty$ with exponential accuracy:

$$\frac{\sigma}{\sin(\pi\nu)} \frac{u^{\frac{\nu}{2}} I_{-\nu}(2\sqrt{u})}{u^{-\frac{\nu}{2}} I_{\nu}(2\sqrt{u})} \rightarrow \frac{\sigma}{\sin(\pi\nu)} \left(u^{\nu} + O(\exp(-2u))\right), u = -R^2 t \rightarrow \infty \quad (72)$$

Clearly, it has to be faster than any power, otherwise one would not be able to fix the pole positions and residues. One can arrange such shifts in pole position that the sum of poles would change only by t^{-n} with arbitrary large n if we take many poles and conspire their shifts so that first $n-1$ terms of expansion in inverse powers of t of these pole terms will all vanish. However, with exponential accuracy we have in M-limit, there is no room for the pole shifting. Should one forget about such requirement, one could play with nice phenomenological "models" like $\psi(t)$ with linearly rising spectrum of masses. We do not have a luxury of choosing the models which fit theoretical expectations or experimental data. We derive mass spectrum from planar graphs by regularizing them in the

IR region. As QED did in its own time, true perturbation theory must be able to remove all cutoffs in observable quantities after certain renormalization. This is the goal of the next Section.

5 Dimensional regularization of LargeN QCD and Pade Theory

In case of perturbative QCD within dimensional regularization we simply have in above formulas

$$\begin{aligned} g_k &= G_k(\epsilon)\lambda^k, \\ \mu_k &= k\epsilon, \\ \nu &= \Delta - 2 + \epsilon/2, \end{aligned} \tag{73}$$

where $d = 4 - \epsilon$ is space-time dimension and Δ is scaling dimension of the conformal operator. As usual, λ is t'Hoofts constant in dimensional regularization, having dimension of $[m]^\epsilon$.Coefficients $G_k(\epsilon)$ (sum of all planar graphs of order k in coupling constant) are some calculable functions of ϵ which we need in terms of Laurent expansion in inverse powers of ϵ starting with ϵ^{-k-1}. The limit $\epsilon \to 0$ can be performed in observables after dimensional transmutation (see later).

First of all we are pleased to note that in large order k of perturbation expansion the ratios of Γ functions in front of $G_k(\epsilon)$ in our sums over k decrease as factorials of k so that we have absolute convergence of regularized perturbation expansion at any finite ϵ. As for convergence in four dimensions, it cannot be proven so easily, because it requires taking the limit $\epsilon \to +0$ but let me make a following simple observation. The number of planar graphs grows only exponentially, as is well known, so that the divergence of perturbation expansion in the Large N QCD has to do with growth of the Feynman integrals at higher order k. As 't Hooft argued long ago, there are so called renormalons, namely condensing singularities in complex plane of coupling constant λ . These singularities reflect precisely the discrete spectrum of masses which are poles at $p^2/\mu^2 = t_n$ where μ is physical mass scale. This μ behaves as $\exp(-C/\lambda)$ at small λ so that growing mass spectrum corresponds to singularities in λ plane condensing to the origin as $\frac{C}{\log(t_n/p^2)}$. These singularities make the origin an essential singularity point, eliminating any hopes for convergence of planar graph expansion for 2-point functions. At finite ϵ the renormalon argument still works, as in this case μ behaves as $\lambda^{\frac{1}{\epsilon}}$ so that singularities in complex λ plane condense to the origin as $\left(\frac{p^2}{t_n}\right)^{\frac{\epsilon}{2}}$.

But the whole point of Pade regularization is to solve this renormalon problem. We restore the correct analytic properties of 2-point function with infinite number of growing masses. The Pade approximant in every order of our expansion has discrete spectrum, with density being equal to sum of delta functions.

The observables we are expanding in perturbation series, do not depend on momentum variables– these are precisely these discrete masses we are expanding in powers of effective coupling constant. The renormalon arguments simply do not apply once we made spectrum discrete. At finite ϵ it is obvious because of extra factorial convergence we obtain in our expansion. At $\epsilon \to 0$ we cannot prove it but the common sense says that observable quantities have no singularities at $\epsilon \to 0$, so they should at $\epsilon = 0$ be close enough to what they are at $\epsilon = 0.1$. Otherwise we will keep ϵ small and finite , compute observables and numerically extrapolate to $\epsilon = 0$. This is of course just a joke – perturbation expansion allows to set $\epsilon = 0$ in every order after coupling renormalization.

One may wonder how could we have avoided the simple fact that QCD mass scale has $\exp(-C/\lambda)$ or $\lambda^{\frac{1}{\epsilon}}$ singularity at small coupling constant. How can we expand in coupling constant at all? The answer is simple: at finite IR cutoff R there are no singularity in the mass spectrum as a function of coupling constant. Particles are confined in a (fifth dimension) box as part of regularization, so they have discrete rising spectrum even without interaction. The singularities would come back in the limit $R \to \infty$ but we take another effective coupling α which goes to 1 in the IR limit without any singularities.

As physicists we all know that quarks are not very strongly interacting inside mesons, effective coupling is small enough to make free quark picture close to reality. It is just the limitations of modern QFT technology which prevent us from computing masses and other observables of Large N QCD with weakly interacting quarks at confinement scale. What we are suggesting here is the way around these limitations of perturbative QCD.

I vividly remember discussion of my old paper with young Ed Witten in 1976 in Boston. He quietly listened to my excited presentation of what I perceived as a solution of the IR divergency problem in QCD and he asked only one question: "What is a physical meaning of your infrared cutoff R ?". I could not have answered that question at that time, so I started waving hands. "Well, it preserves the positivity and correct analytic properties of QCD and its space-time symmetries, so there must be some physical interpretation, but honestly I do not know it. It must be a box in some extra dimension"–said I without any idea what I was talking about.

This was when the sales of my theory went down. Nobody needed computational method without compelling physical picture, even if this was as fictitious as a string in some imaginary space. This reflects the basic laws of psychology: people care about stories more than they care about material things in life.

I still cannot answer Ed's question today, though some fantastic physical picture have emerged with AdS/CFT analogy. We now say that R is a size of the box in fifth coordinate in Ads space. This picture, unfortunately, is still incomplete. No AdS model was found for large N QCD, only for some SUSY models with degenerate set of RG equations (some coupling constants do not run and remain as free parameters). I think that there must be more general physical picture, without SUSY and CFT. While everybody keeps looking for this general interpretation, we still can take M-limit of matrix Pade approximant as a mathematical definition of Large N QCD and study the regularized perturbation expansion in a hope to understand its physical meaning.

6 Infrared regularization and effective coupling constant

Let us consider pure QCD in glueball sector, which decouples at $N_c = \infty$. The challenging problem of chiral symmetry breaking is not present here, so this the place to start testing new perturbation expansion. All masses are expected to be finite in this sector, so we can use any mass as a physical scale in renormalization scheme. The most fundamental and simple quantity is the 2-point function of stress-energy tensor.

$$\Pi_{\mu\nu}^{\alpha\beta}(p) = \int d^4x e^{ipx} \left\langle \Theta_{\mu\nu}(0), \Theta^{\alpha\beta}(x) \right\rangle , \tag{74}$$

$$\Theta_{\mu\nu}(x) = \frac{1}{N_c} \operatorname{Tr}(F_{\mu\alpha}(x) F_{\nu\alpha}(x)) - trace. \tag{75}$$

Let us consider the family of poles of Pade approximant with quantum numbers of $\Theta_{\mu\nu}(x)$ in 68 (so called vacuum Regge trajectory) :

$$m_n^2 = R^{-2} \exp\left(f_n\left(\lambda_R\right)\right) \tag{76}$$

where the function $f_n\left(\lambda\right)$ can be obtained perturbatively from above matrix equation for $Q(t)$. We denote the running coupling constant at scale R^{-1} as λ_R, assuming that the UV regularization is already removed, so that $\epsilon = 0$.

Let us look at the structure of the relation for m_n^2 . There is one important fact in Pade theory: it always overestimates the masses. In other words, at any fixed Λ every mass monotonously decreases with N. Taking the liberty of assuming that the same property holds in the M-limit, where $R = N/\sqrt{\Lambda}$ remains finite after setting $\Lambda = \infty$, we conclude that $\log(m_n^2)$ monotonously decreases with scale $\log(R^2)$. In terms of β-function this inequality reads:

$$-\beta(\lambda) f_n^{'}\left(\lambda\right) < 1 \tag{77}$$

At small R this decrease is trivial, as the factor R^{-2} is the fastest changing factor in perturbative region. The masses of free particles in a box size R decrease as $1/R$ when the box grows. However, we expect that with increase of R the growth of effective coupling constant λ_R will stop this decrease so that asymptotically at large R the masses approach finite limits from above. In that limit the above inequality turns into equality

$$-\beta(\lambda) f_n^{'}\left(\lambda\right) \to 1^{-}. \tag{78}$$

In terms of function $f_n\left(\lambda_R\right)$ this mean that it goes to $+\infty$ as $\log(R^2)$. In order to interpolate between weak and strong coupling regions let us generalize this formula by applying Legendre transform:

$$\log(\frac{m_n^2}{\mu^2}) = \lim_{\alpha\to 1^-} \Phi_n(\alpha), \tag{79}$$

$$\Phi_n(\alpha) = \min_R \left(f_n\left(\lambda_R\right) - \alpha^2 \log(\mu^2 R^2)\right), \tag{80}$$

$$\Phi_n^{'}(\alpha) = -2\alpha \log(\mu^2 R^2). \tag{81}$$

At any $\alpha < 1$ the RHS of above formula goes to $+\infty$ as $(1-\alpha^2)\log(R^2)$ at $R \to \infty$. On the other hand, at small R it also goes to $+\infty$ as $-\alpha^2\log(R^2)$. Therefore there must be at least one minimum in between. In virtue of the Pade theorems about monotonic mass decrease we may expect that there is only one minimum. The idea behind the α expansion we proposed in the old paper ([3]) is that there is smooth interpolation between perturbative region of small α and confining theory at $\alpha = 1$.

The equation for the minimum in 79 involves the β-function:

$$-\beta(\lambda) f_n^{'}(\lambda) = \alpha^2. \tag{82}$$

As the β-function starts with $-a\lambda^2$ with positive a we get:

$$\lambda_n(\alpha) \to \frac{\alpha}{\sqrt{a f_n^{'}(0)}}. \tag{83}$$

So, in general, the Legendre transform $\Phi_n(\alpha)$ starts with some constant at $\alpha = 0$ then grows, reaches its maximum at some point where $\mu R = 1$, then starts decreasing and reaches the limit at $\alpha = 1$. This is much smoother behavior than the R-dependence of original function $f_n\left(\lambda_R\right) - \log(R^2)$, which starts at $+\infty$ as $-\log(R^2)$ at small R then decreases and reaches the limit at $R \to \infty$. This was the purpose of the Legendre transform to achieve dimensional transmutation and obtain smoother behavior. No finite order of perturbation expansion could have reproduced the confining asymptotics at $R \to \infty$. On the contrary, within the α-expansion starting with the second order, we can have expected properties: linear growth at small α then maximum at finite α then descend to another limit at $\alpha = 1$. The drama was totally eliminated from confinement story by means of this Legendre transform.

The first 4 coefficients of α-expansion for all basic trajectories (vector, scalar, vacuum) were computed in my old paper ([2]). For the operator with $n-2$ extra derivatives

$$\Theta^{(n-2)}_{\mu\nu\ldots}(x) = \frac{1}{N_c}\,\mathrm{Tr}(F_{\mu\alpha}(x)\nabla\ldots\nabla F_{\nu\alpha}(x)) - traces.$$

the coefficients $f_n^{'}(0)$ are proportional to anomalous dimensions in leading order

$$\gamma_n^{'} = \frac{6}{11}\left[\frac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_{j=2}^{n}\frac{1}{j}\right] \tag{84}$$

which is positive for $n > 2$ and vanish at $n = 2$ in virtue of conservation of energy-momentum tensor. This means that lowest mass cannot be taken as a definition of physical scale in this procedure. We can take the next one, $n = 3$, use the relation between effective coupling and α and compute remaining masses in terms of this particular α corresponding to $n = 3$.

$$\log\left(\frac{m_n^2}{m_3^3}\right) = f_n(\lambda) - f_3(\lambda), \tag{85}$$

$$-\beta(\lambda) f_3^{'}(\lambda) = \alpha^2. \tag{86}$$

This expansion is constructed so that all terms are universal and calculable, with zeroth term having good resemblance to reality, as we know from the old paper as well as its rebirthing with AdS/CFT models. One may argue that there are infinitely many ways to build such an expansion and we agree with that. But the same can be said about Wilson's ϵ expansion, which interpolated between 4 and $4 - \epsilon = 3$ dimensions in the phase transition theory. There are many ways of analytic continuation into fractional dimension of space, but the most natural one chosen by Wilson turned out to work well in practice. It remains to be seen whether this α expansion will be as lucky.

7 Conclusion

The approach we revived in this paper is opposite to modern quest for String Theory solution of Large N QCD. Instead of finding peculiar string theory equivalent to perturbative QCD in the UV region we postulate existence of such string theory and are studying its properties without a luxury of some local 2D field theory as a dual definition. We rather introduce most general free meson theory in 4D with unknown masses and coupling constants to QCD gauge invariant composite fields and derive explicit equations for these masses and coupling constants.

The underlying idea is that perturbative QCD is very close to reality, as the effective coupling constant at hadron scale is small. It is just the IR divergencies which prevent us from perturbative calculations of mass ratios. Once we regularize the perturbation theory we can expect rapid convergence, because the planar graphs grow only a power of order of perturbation expansion. Renormalons look like a paper tiger: they disappear after regularization. On top of general analytic arguments there are extra factorials in denominator of our expansion, arising from transformation from perturbative QCD to the mass spectral operator.

The educated reader (of older generation) may say: wait a minute, but where are SVZ vacuum condensates? My answer is simple: they are no longer needed as we do not truncate the perturbation expansion. We rather argue that this expansion converge. SVZ condensates were designed to phenomenologically describe renormalons: small momentum regions in Feynman integrals of higher order, behaving as powers of QCD mass scale. These powerlike terms balanced the decrease of masses as function of R and SVZ obtained good agreement with experiment by matching one-loop QCD with sum of pole terms at the scale determined by the vacuum condensate terms.

Once we add all terms of perturbation expansion there is no need for these matching procedures. The planar graphs are both UV and IR finite in our regularized perturbation expansion, and this expansion converges as double factorial in denominator, similar to Bessel functions.

One last word about notorious Pade approximation. I hope the reader understands by now that there are no approximations involved in our theory. This is regularization rather than approximation. We impose correct analytic properties and symmetries of large N QCD beyond perturbation theory by first using Matrix Pade approximation in Hilbert space and then taking M limit when approximation becomes realistic in a sense that it has all the desired properties including infinite mass spectrum.

Note that conformal symmetry of the leading order was absolutely essential to obtain calculable perturbation expansion. It is because of conformal symmetry our leading order Q matrix becomes block-diagonal in space of conformal tensors, so that we can invert it in Pade equations. The conformal group representation of meson states is broken down to general Lorentz group representation in the second order in our perturbation expansion.

The interpolation to physical limit when the approximation disappears completely is similar to UV regularization of QED half century ago. The cutoff enters only in logarithms, effectively renormalizing running coupling constant. By renormalization, using Legendre transform, we trade this cutoff for the effective coupling constant α which should be set to 1 in the end.

I no longer hope that anyone will pick this theory and actually carry out these computations of Meson spectrum. I will do my best to finish this work myself. The terms of this renormalized perturbation expansion are all calculable, and we presented explicit formulas in this paper. The hardest problem is to compute ordinary planar graphs for 2-point functions with dimensional regularization. I am aware of 3 and 4 loop calculations, but this may not be enough. Maybe some recent progress in perturbative calculations in SUSY Yang Mills theory can help here?

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