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On the Nonrelativistic Purely Magnetic Supersymmetric Pauli Operator (for the 2D particles with spin 1/2):

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Introduction: Completely Integrable PDE Dynamical Systems and Linear Operators:

The typical 1D Case: KdV is equivalent to the Lax Pair

$$u_t = 6uu_x - u_{xxx} \to dL/dt = [A, L],$$
$$L = -\partial_x^2 + u(x,t)$$
$$A = \partial_x^3 + 3/2(u\partial_x + \partial_x u)$$

There are also (3/2)D Cases like KP

The 2D analog of Lax Pair is:

$$dL/dt = [A, L] + fL$$

It was found by Manakov and used since 1976 by Novikov's Group DKN:

$$L = -\Delta + U\partial_x + V\partial_y + V(x, y)$$

There are different possibilities for the operator A: It was chosen of the 3d order to get physically interesting Nonlinear Systems. We choose it now of the order two for the second kind applications (below).

Two different ways to use this presentation are known:

I. From the Spectral Theory of the operator L to the Solutions of Nonlinear System (The Inverse Scattering Transform for the Solitons and Algebro-Geometric Solutions for the Periodic Problem).

II. From Nonlinear Systems to the Nontrivial Exactly Solvable Quantum Systems: (The Algebro-Geometric or Finite-Gap Periodic 1D Schrodinger Operators and 2D Scalar Operators with One Selected Energy Level $\epsilon = \epsilon_0$).

Principle: Is it possible to solve efficiently a inverse spectral problem (inverse scattering problem) for a linear operator?

The answer is yes iff there exists an integrable hierarchy associated with this problem. This hierarchy generates the symmetries of this problem.

Special case: the problem of reductions.
Examples: Pure electric 2D Schrodinger operators, pure magnetic 2D Schrodinger operators. Reductions can be described in terms of spectral data if they are compatible with associated hierarchies.

Our Goal now is to extend these ideas to the special case of the 2D Purely Magnetic Pauli Operator L^P (spin 1/2). There is something here deserving todays discussion.

In 1979-1980 three groups of authors studied the ground level using "The Factozization Property" of the 2D purely Magnetic Pauli Operator written in the Lorenz gauge $A_{1x} + A_{2y} = 0$ with $A_1 = i\Phi_y, A_2 = -i\Phi_x$: (Avron-Seiler[AS], Aharonov-Casher[AC], Dubrovin and Novikov [DN]:

$L^P = L^+ \oplus L^-,$

and

$-L^{\pm} = (\partial_x + i\Phi_y)^2 + (\partial_y - i\Phi_x)^2 \pm \Delta\Phi$

It acts in the space of vector-function

$\Psi = (\Psi^+, \Psi^-).$

Take $Q = \partial_z + A_z, B = \Delta\Phi$, $A_z = A_1 - iA_2 = -\Phi_x + i\Phi_y$, magnetic field $B = \Delta\Phi$ in our units.

The Operators $L^{\pm}$ are Factorized

$L^+ = QQ^+, L^- = Q^+Q$

The most interesting classes of magnetic fields are [AC] and [DN].

1.AC: Rapidly decreasing fields, $|[B]| = |\int_{R^2} B dx dy| < \infty$. Ground states form a finite-dimensional space of dimension $m \in Z, m \leq [B] < m + 1$

2.DN: Periodic fields with integer flux $\neq 0$ through the elementary cell $0 \neq \int_{cell} B dx dy = m \in Z$.

The ground states form an infinite dimensional subspace in the Hilbert Space $L_2(R^2)$ isomorphic to the Landau level.

Remark:

"Generic" operators and their topology in the space of quasimomenta, in particular Chern numbers of the transversal dispersion relations, were studied in 1980-81 by Novikov and A.Lyskova as a continuation of this work. It was partly rediscovered by physicists of the Thouless group few years later after the experimental discovery of the famous "Integral Quantum Hall Fenomenon".

In the cases AC and DN all ground states are the Instantons belonging to one spin-sector only:

a. They satisfy to the 1st order equations $Q^{+}\psi = 0$ for the case $[B] > 0$ and $Q\psi = 0$ for the case $[B] < 0$. It is a simple prototype of the self-duality equation.

b. They belong to the Hilbert Space $L_2(R^2)$

The operator

$$S : (\Psi^+, \Psi^-) \to (0, Q^+\Psi^+)$$

is a "Super-Symmetry" for L^P. Here $S^2 = 0, SL^P = L^P S, S^* L^P = L^P S^*$. The "adjoint" supersymmetry operator is

$$S^* : (\Psi^+, \Psi^-) \to (Q\Psi^-, 0)$$
$$SS^* + S^* S = L^P.$$

It is special case of the "Laplace Transformations" known since XVIII Century. It implies that all higher levels are 2-degenerate (the ground level is ∞-degenerate).

How to unify this technic with methods of The Soliton Theory? There are Difficulties here:

1. In the Algebro-Geometric Theory of the 2D Second Order Scalar Schrodinger Operators and Corresponding Soliton Hierarchies (started in 1976 by Manakov[M] and Dubrovin-Krichever-Novikov) Magnetic Field is Always Topologically Trivial: $\text{Flux} = 0$

2.No Reduction was known leading to the Factorized operators. The case of zero flux was unsolved. We use Nonlinear Systems.

Why Nonlinear Systems are useful here? How to use them? The existence of time-invariant reduction is much easier to see on the level of equation than in the Spectral Theory of operator. Many years ago Novikov made observation: Only time-invariant reductions have good solution of the Inverse Scattering Problem.

Consider a very first Manakov's System $L_t = [H, L] + fL$ where L, H are the second order operators: $L = \partial_x \partial_y + G\partial_y + S, H = \Delta + F\partial_y + A$. Konopelchenko pointed out in 1988 that the reduction $S = 0$ is time-invariant. It looks like 2D analog of the famous Burgers system. Make replacement $x, y \to z, \bar{z}$ to get elliptic operators interesting for us.

Our Conclusion: Corresponding Inverse Problem can be effectively solved: The Solution is:

Take Riemann Surface (the Complex Fermi Curve) splitted into nonsingular pieces $\Gamma = \Gamma' \cup \Gamma''$ with genuses g', g''. They cross each other $P_j = Q_j$, $P_j \in \Gamma'', Q_j \in \Gamma'$, $j = 0, ..., l$.
Take infinities, i.e. 2 points ∞_1 in Γ',∞_2 in Γ'', with local parameters $1/k', 1/k''$. Our vector-function $\tilde{\psi} = (\psi', \psi'')$ is meromorphic outside of infinities

$$\psi' \sim c(x,y)e^{k'\bar{z}}(1 + O(k'^{-1})),$$
$$\psi'' \sim e^{k''z}(1 + O(k''^{-1})),$$

they have constant poles D', D'' consisting of $g' + l, g''$ points, not crossing infinities and intersection points, $z = x + iy$.

From this Data we calculate a psi-function $\tilde{\psi} = (\psi', \psi'')$ and operator $\tilde{L} = \Delta + G\partial_{\bar{z}}$ with $S = 0$ such that $\tilde{L}\psi' = \tilde{L}\psi'' = 0$. We can explicitly calculate them.

For the self-adjoint case $g' = g''$, and reduced Data can be described.
The reduced Data after the non-unitary gauge transformation

$$L = c^{-1/2}\tilde{L}c^{1/2},\ \tilde{\psi}c^{-1/2} = \psi$$

generate an operator $L = QQ^{+}$.
Taking $L^{+} = L$ and $L^{-} = Q^{+}Q$, we construct a Purely Magnetic Pauli Operator

$$L^{P} = QQ^{+} \oplus Q^{+}Q.$$

The Magnetic Field is real $B = 1/2\Delta \log c$, periodic or quasiperiodic and **Topologically Trivial**. It is nonsingular if $c \neq 0$, so the operator is self-adjoint in this case.

Easy to find ground states here:

Take $\psi_0 = c^{1/2}$ in the first spin-sector because $Q^+\psi_0 = 0$.

Take $\phi_0 = c^{-1/2}$ in the second sector because $Q\phi_0 = 0$.

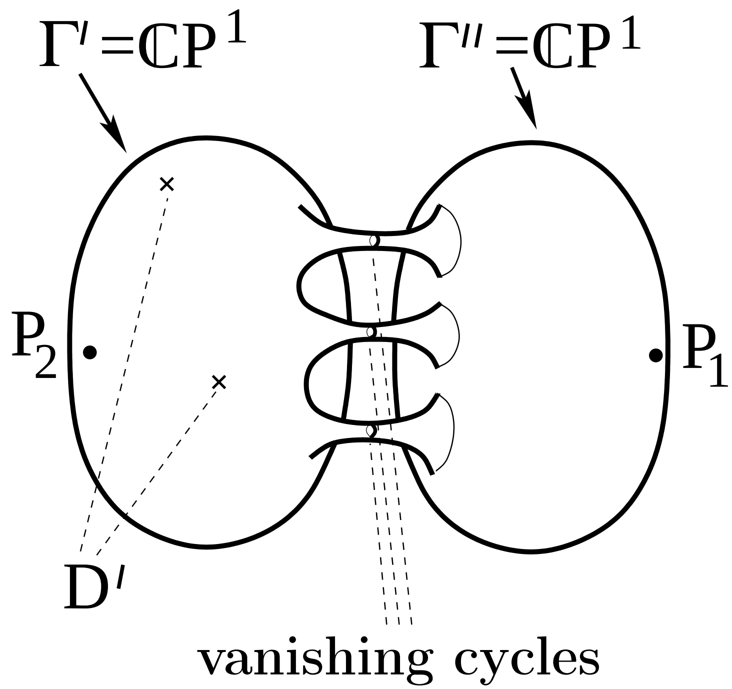
For periodic $c \neq 0$ we have two periodic ground state functions located in both sectors. They obviously are at the bottom of the CONTINUOUS SPECTRUM.

We obtained full description of all complex nonsingular Bloch functions of the ground level.

Is there any relationship of this functions with ground states found in [DN] in 1980 for the topologically nontrivial magnetic field? The cases $g = 0, 1$ below will lead to solution of that problem.

The Case of Genus zero

Fig 1



We take $l+1$ intersection points presented as $k' = k_s$ and $k'' = p_s$ in Γ', Γ'', and divisor $D' = (a_1, ..., a_l)$ of degree l in Γ'. We have $\Psi = e^{k'\bar{z}} \frac{w_0 k'^l + ... + w_l}{(k' - a_1)...(k' - a_l)}, \Psi|_{k'=k_s} = e^{p_s z}$. As we can see, $c = w_0$.

So $c = \sum_{s=0}^{l} \kappa_s e^{W_s(z,\bar{z})}$, where W_s is a linear form. All complex coefficients are possible.

$W_s = \alpha_s x + \beta_s y, (\alpha_s, \beta_s) \in C^2_W$. Transformation $c \to c' = ce^{\gamma + \alpha x + \beta y}$ leads to the gauge equivalent operator (the same magnetic field)

There exist 3 types of Real Solutions:

1.Purely Exponential Positive Case (The Lump-type fields”) $\kappa_s > 0, (\alpha_s, \beta_s) \in R$.

2.Periodic Trigonometric Real Case. It will be considered below jointly with the case $g = 1$

3.The mixed case. It can be realized only if its ”dominating part” belongs to the case 1. We will not discuss it.

The case 1. Let "the Tropical Sum" of the forms in the set $\{W\}$ is nonnegative $I'_{\{W\}}(\phi) =$ $= \max_s(\alpha_s \cos\phi + \beta_s \sin\phi) \geq 0$.

Then $c^{-1/2}$ is bounded in R^2

For the angles $I'_{\{W\}}(\phi) > 0$ we have a rapid decay

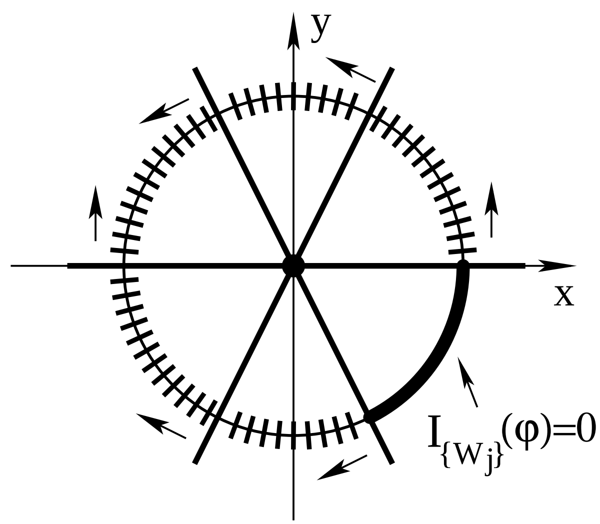
$c^{-1/2} \to 0, R \to \infty,$

Let

$I(\phi) = \max\{I'(\phi), 0\}$

Fig 2a

$$c=e^{y}+e^{y-2x}+e^{-y-2x}$$

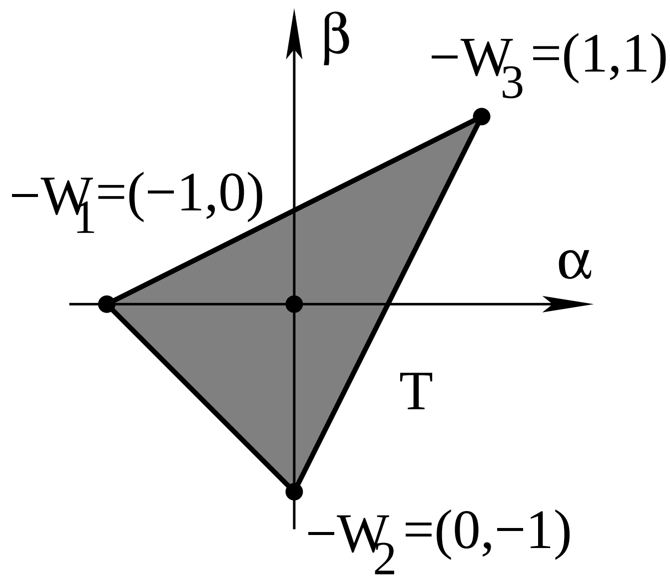


In every class $c' \in ce^W, W' \in R^2_W$, the set of representatives c' with nonnegative $I = I'_{\{W'\}}(\phi) \geq 0$ forms a convex polytop $\bar{T}_c$. Its inner part $T_c \subset \bar{T}_c$ consists of all c' such that $I_{\{W'\}} > 0$. Open part T_c is always nonempty for $l > 2$. $\bar{T}_c$ is nonempty for $l > 1$. (see Fig 2b for $l = 3$)

Fig 2b

b) $\mathbb{R}^2 = (\alpha, \beta)$

$W_j = \alpha_j x + \beta_j y$



Here $e^y + e^x + e^{-y-x} = c$

Magnetic field is decaying for $R \to \infty$ except some selected angles, it is a Lump Type Field analogous to the KP "Lump Potentials".A linear sum under the $1/2\Delta\log()$ reflects linearization of the Burgers Hierarchy in the variable c.

$[B] = \int\int_{D_R^2} B dx dy =$

$= -1/2R \oint_{S^1} I_{\{W\}}(\phi)d\phi + O(R^{-1})$ All points in T_c define ground states in the Hilbert Space $L_2(R^2)$. The boundary points define the bottom of continuous spectrum.

The Periodic Problem. Let lattice in R^2 be rectangular and $z=x+iy$. For every real periodic function c we can define a whole family of "possible" meromorphic Bloch functions

$\psi''_{ext,\pm}=f(z)(\sqrt{c})^{\pm}e^{uz-\zeta(p)z}\sigma(z+p+R)/\sigma(z+R)$ where $f(z)$ is an arbitrary elliptic function. We have $L^{+}\psi''_{ext,-}=Q^{+}\psi''_{ext,-}=0$. For anti-holomorphic case $z\to\bar{z}$ and $L^{-}\psi''_{ext,+}=Q\psi''_{ext,+}=0$

Let $c\neq 0$. We need only nonsingular functions, so our manifold is $u\in CP^1=\Gamma''$ and ψ''_{+} is equal to $e^{uz}\sqrt{c}$ or

$e^{u\bar{z}}\sqrt{c}$.

Let c have an isotropic zero.

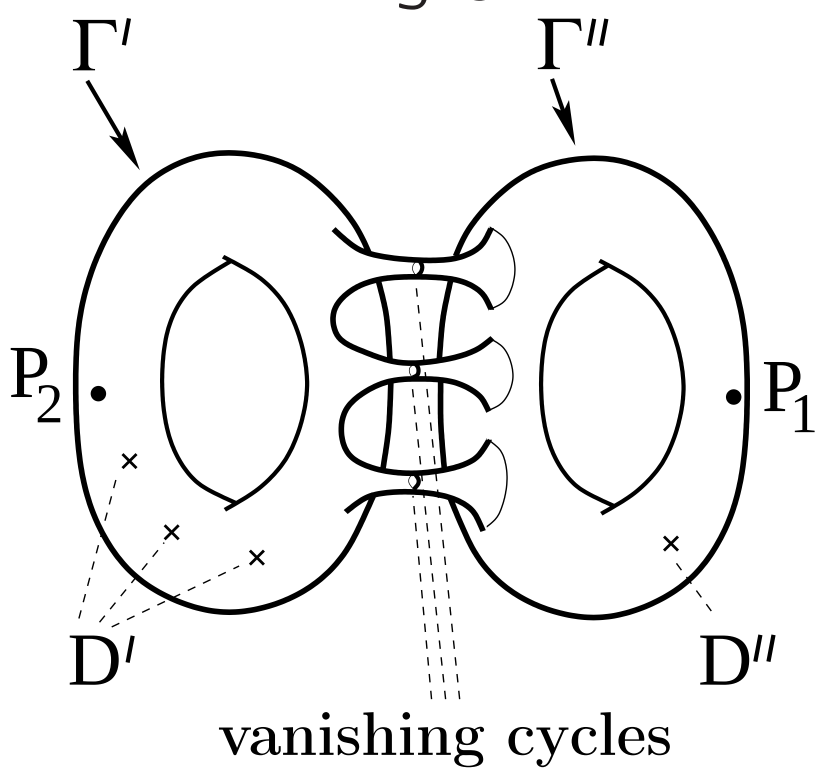
First: The nonsingular family ψ'' became larger, with manifold $M^2 = CP^1 \times \Gamma$ where Γ is elliptic.

Second: The function c for L^+ should be replaced by $c' = 1/c$ for L^-. To calculate ψ' for both sectors we should extend our formulas to all periodic functions.

Magnetic field became sum of smooth field and Aharonov-Bohm δ-term with integer flux.

The case of genus 1.

Fig 3



We take elliptic curve $\Gamma' = \Gamma'' = C/\Lambda$ with euclidean local parameters k, p (the point 0 is "infinity"), periods $1, 2i\omega \in iR$, n intersection points $Q_0, Q_1, ..., Q_n \in \Gamma'$ and $R_0, ..., R_n \in \Gamma''$. Divisors $D' = (P_1, ..., P_n), D'' = P$ have degree $n + 1, 1$ correspondingly. We have $\psi' = e^{-\bar{z}\zeta(k)} \frac{\prod_s \sigma(k-Q_s)}{\prod_l \sigma(k+P_l)} \times$
$\times (\sum_j w_j \frac{\sigma(k+\bar{z}+\tilde{P}+\tilde{Q}-Q_j)}{\sigma(k-Q_j)})$. Here $\tilde{P} = P_1 + ... + P_n, \tilde{Q} = Q_0 + ... + Q_n$, sum as in C

$\psi'' = e^{-z\zeta(p)}\sigma(p+z+P)/(\sigma(z+P)\sigma(p+P))$,
$\psi'(Q_s) = \psi''(R_s)$.
All singularity of the quantity c disappear after multiplication $\tilde{c} = c\sigma(\bar{z}+\tilde{Q}+\tilde{P})\sigma(z+P)$.
Take $n = 1, Q_0 = -Q_1, R_0 = Q_1, R_1 = Q_0$ and solution to the equation
$\omega\zeta(Q_0) = \eta_1 Q_0$ We have $P = \tilde{Q} + \tilde{P}$ in this case, so $-1/2\Delta|\sigma|^2 = -2\pi\delta(z)$.

So our Conclusion based on the case $g = 1$ is:

The magnetic field $\tilde{B} = -1/2\Delta\tilde{c}$ is periodic nonsingular with magnetic flux equal to ONE QUANTUM UNIT. The magnetic field $B = 1/2\Delta c$ is always singular for $g = 1$; it has magnetic flux equal to zero through the elementary cell and δ-singularity in the point P. So this field corresponds to the "Aharonov-Bohm" (AB) situation.

For $g > 1$ number of quantized δ-functions is equal to $k > 1$. Both pieces of the original Riemann surface $\Gamma = \Gamma'' \cup \Gamma'$ are presented in the form of k-sheeted branching covering over elliptic curve $\Gamma'' \rightarrow \Gamma_0$ as it was in the works of Krichever dedicated to the elliptic KP.

Comparison with [DN] shows that the Quantized δ-flux does not affect spectrum in our case.

The complex Bloch-Floquet manifolds (consisting of nonsingular Bloch functions) for the level $\epsilon = 0$ and genus $g = 1$ is $M = M^2 \cup \Gamma'$ with functions ψ' and

$$\psi''_{ext,-} = (1/\sqrt{c})[e^{uz} \times e^{-\zeta(p)z}\sigma(z+p+R)/\sigma(z+R)],$$

$$L^+\psi''_{ext} = L^+\psi' = 0.$$

We did not proved yet that ψ' cannot be extended to the higher dimensional component at the same level, but it is highly probable.

Reconsider now the case $g = 0$ comparing it with $g = 1$.

For $c \neq 0$ and $g = 0$ the Bloch manifold is equal to the union $\Gamma'' \cup \Gamma'$, and both are CP^1;

Let c have an isolated zero (minimum) which is isotropic. Magnetic field became singular, with δ-term. The extended Bloch function can be defined for the operator on the manifold $M^2 = CP^1 \times \Gamma_0$ where Γ_0 is an elliptic curve, $\psi''_{ext,+} = (const(u))e^{p\bar{z}-\zeta(u)\bar{z}}\sigma(\bar{z} + u)\sqrt{c}/\sigma(\bar{z})$.

Our Conclusion is that the periodic case $g = 1$ gives the same result as the special case $g = 0$ where c has an isolated isotropic zero, interchanging sectors $\pm$.

The higher number $k \geq 1$ of isotropic zeroes for $g = 0$ leads to the "higher rank" family of nonsingular Bloch functions $M^{k+1} \cup \Gamma'$. Removing δ-singularities by the singular gauge transformations we get smooth periodic magnetic field like in DN with higher flux.

We know that the algebro-geometric case simply corresponds to the case of trigonometric polynomials. We take rectangular lattice in the plane x, y. Following relation is true $Q^{+}\psi' = M(k)\sqrt{c}e^{\bar{z}k}$ Now we choose normalization of ψ' such that $M(k) = 1$

It is "the Sypersymmetry Operator" in periodic case.

Let us extend our theory to the "infinite" trigonometric series

We use for that the formula

$$\psi' = k \sum_j [\kappa_j e^{p_j z - k_j \bar{z}}/(k - k_j)] e^{k\bar{z}}$$

for this new normalization where k_j are the lattice points. Here

$$\sum_j \kappa_j e^{p_j z - k_j \bar{z}} = c$$

Apply this result to the $c \to c' = 1/c$ which is an infinite trigonometric series. It gives us a function ψ' for the second component L^- of the Pauli operator.

Problem: The component Γ' of the Bloch manifold does not affect the ordinary spectrum in the Hilbert space of functions in the whole plane R^2. Can we use it for solving physically meaningful self-adjoint boundary problems?

New results dedicated to this problem will be published soon by the authors.